CONVERGENCE THEOREMS OF A PSEUDO-NONEXPANSIVE MAPPING AND A MAXIMAL MONOTONE OPERATOR IN A BANACH SPACE (Nonlinear Analysis and Convex Analysis)

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1. Preliminaries

Let E be a smooth Banach space with a norm $\|\cdot\|$ and let C be a nonempty, closed and convex subset of E. We use the following bifunction $V(\cdot,\cdot)$ studied by Alber [1], and Kamimura and Takahashi [11]. Let $V(\cdot,\cdot) : E \times E \to [0,\infty)$ be defined by $V(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for any $x, y \in E$, where $\langle\cdot,\cdot\rangle$ stands for the duality pair and J is the normalized duality mapping. Note that the duality mapping is single-valued in a smooth Banach space (see [21]). From the definition of $V(\cdot,\cdot)$ the following properties are trivial:

Lemma 1.1. *(a) For all $x, y, z \in E$,*

$$V(x,y) \leq V(x,y) + V(y,z) = V(x,z) - 2\langle x - y, Jy - Jz\rangle .$$

(b) If a sequence $\{x_n\} \subset E$ satisfies $\lim_{n\to\infty} V(x_n, w) < \infty$ for some $w \in E$, then $\{x_n\}$ is bounded.

Let $F(T)$ be the fixed points set of T. Ibaraki and Takahashi defined a generalized nonexpansive mapping in a Banach space (see [10]).

Definition 1. A mapping $T : C \to C$ is said to be generalized nonexpansive if $F(T) \neq \emptyset$ and $V(Tx,p) \leq V(x,p)$ for all $x \in C$ and $p \in F(T)$.

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if for all $x \in E$ and $t \geq 0$,

$$R(Rx + t(x - Rx)) = Rx.$$

A mapping $R : E \to D$ is called a retraction if $Rx = x$ for all $x \in D$ (see [6]). It is known that a generalized nonexpansive and sunny retraction of E onto D is uniquely determined if E is a smooth and strictly convex Banach space (cf. [18]). Ibaraki and Takahashi proved the following results in [10].

Lemma 1.2. (cf. [10]) *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then there exists a sunny and generalized nonexpansive retraction on $F(T)$.*

A generalized resolvent J_r of a maximal monotone operator $B \subset E^* \times E$ is defined by $J_r = (I + rBJ)^{-1}$ for any real number $r > 0$. It is well-known that $J_r : E \to E$ is single-valued if E is reflexive, smooth and strictly convex (see [9]). From Lemma 1.1 (a), the following proposition is shown.

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Proposition 1.1. *(a) If a sunny retraction R is generalized nonexpansive, then R satisfies*

$$V(x, Rx) + V(Rx, y) = V(x, y) - 2\langle x - Rx, JRx - Jy\rangle \leq V(x, y), \quad \text{for all } x, y \in D. \tag{1}$$

(b) For each $r > 0$, a generalized resolvent J_r satisfies

$$V(x, J_r x) + V(J_r x, p) \leq V(x, p) \quad \text{for all } x \in E \text{ and } p \in F(J_r). \tag{2}$$

Remark 1. The property in Proposition 1.1 (b) means that J_r is generalized nonexpansive for any $r > 0$.

2. Main results

By using the properties of generalized nonexpansive mappings, we show strong convergence theorems for finding fixed points of a generalized nonexpansive mapping and zeroes of a maximal monotone operator.

Theorem 2.1. [14] *Let E be a reflexive, smooth and strictly convex Banach space, and let $\{T_n\}_{n\in\mathbb{N}}$ be a family of generalized nonexpansive mappings. Suppose that $\cap_{n\in\mathbb{N}} F(T_n) = F \neq \emptyset$ and that R is a sunny and generalized nonexpansive retraction from E to F. Let a sequence $\{x_n\}$ be defined as follows: For any $x_1 = x \in E$,*

$$x_{n+1} = RT_n x_n \quad \text{for any } n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to a point x^* in F.

Theorem 2.2. [14] *Let E be a reflexive, smooth and strictly convex Banach space. Let $T : E \to E$ be a generalized nonexpansive and let $B \subset E^* \times E$ be a maximal monotone operator. Suppose that $F(T) \cap (BJ)^{-1}(0) \neq \emptyset$ and that R is a sunny and generalized nonexpansive retraction from E to $F = F(T) \cap (BJ)^{-1}(0)$. Let an iterative sequence $\{x_n\}$ be defined as follows: For any $x = x_1 \in E$,*

$$x_{n+1} = RTJ_{r_n} x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{r_n\}$ is a sequence of nonnegative real numbers. Then, the sequence $\{x_n\}$ converges strongly to a point x^* in $F(T) \cap (BJ)^{-1}(0)$.

Next we define a new pseudo-nonexpansive mapping which is called a V-strongly nonexpansive mapping as follows ([14]).

Definition 2. [14] A mapping $T : C \to E$ is called V-strongly nonexpansive if there exists a constant $\lambda > 0$ such that

$$V(Tx, Ty) \leq V(x, y) - \lambda V((I - T)x, (I - T)y) \tag{3}$$

for all $x, y \in C$, where I is the identity mapping on E. More explicitly, if (3) holds, T is said to be V-strongly nonexpansive with λ.

It is trivial that a V-strongly nonexpansive mapping is generalized nonexpansive if $F(T) \neq \emptyset$. In [16], Reich introduced a class of strongly nonexpansive mappings which is defined with respect to the Bregmann distance $D(\cdot, \cdot)$ corresponding to a convex continuous function f in a reflexive Banach space E. Let S be a convex subset of E, and $T : S \to S$ be a self-mapping of S. A point p in the closure of S is said to be an asymptotically fixed point of T if S contains a sequence $\{x_n\}$ which converges weakly to p and the sequence $\{x_n - Tx_n\}$ converges strongly to 0.

$\hat{F}(T)$ denotes the asymptotically fixed points set of T. The definition of strongly nonexpansive mappings in a reflexive Banach space E is given as follows.

Definition 3. The Bregman distance corresponding to a function $f : E \to R$ is defined by

$$D(x,y) = f(x) - f(y) - f'(y)(x-y),$$

where f is Gâteaux differentiable and $f'(x)$ stands for the derivative of f at the point x. We say that the mapping T is strongly nonexpansive if $\hat{F}(T) \neq \emptyset$ and

$$D(p,Tx) \leq D(p,x) \quad \text{for all } p \in \hat{F}(T) \text{ and } x \in S, \tag{4}$$

and if it holds that $\lim_{n\to\infty} D(Tx_n, x_n) = 0$ for a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}(D(p,x_n) - D(p,Tx_n)) = 0$ for any $p \in \hat{F}(T)$.

Taking the function $\|\cdot\|^2$ as the convex, continuous and Gâteaux differentiable function f, we obtain the fact that the Bregmann distance $D(\cdot,\cdot)$ coincides with $V(\cdot,\cdot)$. Especially in a Hilbert space, $D(x,y) = V(x,y) = \|x-y\|^2$. We shall recall some nonlinear mappings in a Hilbert space H.

Definition 4. Let C be a nonempty, closed and convex subset of H. A mapping $A : C \to H$ is said to be α-inverse strongly monotone if

$$\alpha \|Tx - Ty\|^2 \leq \langle x-y, Tx-Ty\rangle \tag{5}$$

for all $x, y \in C$.

If $A : H \to H$ is an α-inverse monotone operator, then $T = I - A$ satisfies the following inequality.

$$\langle Ax - Ay, x-y\rangle \leq \|x-y\|^2 - \alpha \|(I-A)x - (I-A)y\|^2 .$$

Therefore, we obtain for an α-inverse strongly monotone A with $\alpha > 0$ that $(I-A)$ is V-strongly nonexpansive with a constant α. Furthermore, we have the following result.

Proposition 2.1. [14] *In a Hilbert space H, the followings hold.*

(a) A firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$.

(b) A V-strongly nonexpansive mapping T with $\hat{F}(T) \neq \emptyset$ is strongly nonexpansive.

In a Banach space, V-strongly nonexpansive mappings have the following properties.

Proposition 2.2. [14] *In a smooth Banach space E, the followings hold.*

(a) For $c \in (-1,1]$, $T = cI$ is V-strongly nonexpansive. For $c = 1$, $T = I$ is V-strongly nonexpansive for any $\lambda > 0$. For $c \in (-1,1)$, $T = cI$ is V-strongly nonexpansive for any $\lambda \in (0, \frac{1+c}{1-c}]$.

(b) If T is V-strongly nonexpansive with λ, then for any $\alpha \in [-1,1]$ with $\alpha \neq 0$, αT is also V-strongly nonexpansive with $\alpha^2\lambda$.

(c) If T is V-strongly nonexpansive with $\lambda \geq 1$, then $A = I - T$ is V-strongly nonexpansive with λ^{-1}.

(d) Suppose that T is V-strongly nonexpansive with λ and that $\alpha \in [-1,1]$ satisfies $\alpha^2\lambda \geq 1$. Then $(I - \alpha T)$ is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Moreover, if $T_\alpha = I - \alpha T$, then

$$V(T_\alpha x, T_\alpha y) \leq V(x,y) - \lambda^{-1} V(Tx, Ty). \tag{6}$$

It is obvious that a V-strongly nonexpansive mapping T is nonexpansive in a Hilbert space. However in Banach spaces, as we will show the following example, a V-strongly nonexpansive mapping T is not necessary nonexpansive even if T is a continuous mapping with a fixed point ([15]).

Example 1. [15] Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $E = l^p(\mathbb{R} \times \mathbb{R})$ be a real Banach space with a norm $\|\cdot\|_p$ defined by

$$\|x\|_p = \{|x_1|^p + |x_2|^p\}^{\frac{1}{p}} \quad \text{for all } x = (x_1, x_2) \in E.$$

Then E is smooth, and the normalized duality mapping J is single-valued. J is given by

$$Jx = \|x\|_p^{2-p} \left(x_1|x_1|^{p-2}, x_2|x_2|^{p-2} \right) \in l^q(\mathbb{R} \times \mathbb{R}) \quad \text{for all } x = (x_1, x_2) \in E.$$

Hence we have for $x, y \in E$ that

$$\begin{aligned} V(x,y) &= \|x\|_p^2 + \|y\|_p^2 - 2\langle x, Jy\rangle \\ &= \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p}\left\{x_1y_1|y_1|^{p-2} + x_2y_2|y_2|^{p-2}\right\}. \end{aligned}$$

We define a mapping $T : E \to E$ as follows:

$$Tx = \begin{cases} x & \text{if } \|x\|_p \leq 1, \\ \frac{1}{\|x\|_p}x & \text{if } \|x\|_p > 1. \end{cases}$$

This example simultaneously give a fact that T is not quasi-nonexpansive for some p. Let $p = \frac{3}{2}$, $x = (0,1) \in F(T)$ and $y = (0.2, 0.95) \in E$, we have that

$$\begin{aligned} \|Tx - Ty\|_p^p &= \|y\|_p^{-p}\{(0.2)^{\frac{3}{2}} + (\|y\|_p - 0.95)^{\frac{3}{2}}\} \\ &> (0.2)^{\frac{3}{2}} + (0.05)^{\frac{3}{2}} = \|x - y\|_p^p. \end{aligned}$$

Finally, we give a convergence theorem for finding common zero points of a maximal monotone operator and a V-strongly nonexpansive mappings.

Theorem 2.3. *Let E be a reflexive, smooth and strictly convex Banach space. Suppose that the duality mapping J of E is weakly sequentially continuous. Let C be a nonempty, closed and convex subset of E. Let $B : E^* \to 2^E$ be a maximal monotone operator and let $J_{r_n} = (I + r_nBJ)^{-1}$ be a generalized resolvent of B for a sequence $\{r_n\} \subset (0, \infty)$. Suppose that $T : C \to E$ is a V-strongly nonexpansive mapping with $\lambda \geq 1$ such that $C_0 = T^{-1}(0) \cap (BJ)^{-1}(0) \neq \emptyset$ and that $R_C : E \to C$ is a sunny and generalized nonexpansive retraction. For an $\alpha \in [-1, 1]$ such that $\alpha^2\lambda \geq 1$, let an iterative sequence $\{x_n\} \subset C$ be defined as follows: for any $x = x_1 \in C$ and $n \in \mathbb{N}$,*

$$\begin{cases} y_n = R_C(I - \alpha T)x_n, \\ x_{n+1} = R_C(\beta_n x + (1 - \beta_n)J_{r_n}y_n), \end{cases} \tag{7}$$

where $\{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy that*

$$\sum_{n\geq 1} \beta_n < \infty \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0. \tag{8}$$

Then, there exists an element $u \in C_0$ *such that*

$$x_n \rightharpoonup u \quad \text{and} \quad R_{C_0}(x_n) \to u. \tag{9}$$

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